Quantum Zeno Effect in the Decoherent Histories

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Abstract

The quantum Zeno effect arises due to frequent observation. That implies the existence of some experimenter and its interaction with the system. In this contribution, we examine what happens for a closed system if one considers a quantum Zeno type of question, namely *what is the probability of a system, remaining always in a particular subspace*. This has implications to the arrival time problem that is also discussed. We employ the decoherent histories approach to quantum theory, as this is the better developed formulation of closed system quantum mechanics, and in particular, dealing with questions that involve time in a non-trivial way. We get a very restrictive decoherence condition, that implies that even if we do introduce an environment, there will be very few cases that we can assign probabilities to these histories, but in those cases, the quantum Zeno effect is still present.

1 Motivation

A remarkable property of quantum mechanics, is the so called quantum Zeno effect [1]. This effect, is that frequent observation slow down the evolution of the state, with the limit of continuous observation leading to "freezing" of the state[1]. This has been experimentally verified. The intuitive explanation, is that the interaction of the observer with the system leads to this apparent paradox. It would therefore be interesting to see whether this effect persists if we consider a closed system. We would try to see what is the probability of a closed system remaining in a particular subspace of its Hilbert space with no external observer. This directly relates to the arrival time problem as well (e.g. [2, 3]). Having said that, we should emphasize that in closed systems, we cannot in general assign probabilities to histories, unless they decohere and it is this property that resolves the apparent paradox that arises.

2 This paper

This contribution is largely based on Ref.[3]. In Section 3 we revise the quantum Zeno effect and the decoherent histories, and introduce a new formula for the restricted propagator that will be of use further. In Section 4.1 we see what probabilities we would get if we had decoherence, that highlights the persistence of the quantum Zeno effect. In Section 4.2 we get the decoherence condition that in Section 5 is stressed how restrictive is by considering the arrival time problem. We conclude in Section 6.

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[1]To be more precise, restriction to a subspace.

3 Introductory material

3.1 Quantum Zeno effect

In standard Copenhagen quantum mechanics, the measurement is represented by projecting the state to a subspace defined by the eigenstates that correspond to the range of eigenvalues of the measured physical quantity. The latter is represented by a self-adjoint operator. The state, otherwise evolves unitarily: $\hat{U}(t) = \exp(-i\hat{H}t)$, where $\hat{H}$ is the Hamiltonian. It is then a mathematical fact, that frequent measurement, of the same quantity (subspace) leads to slow down of the evolution, i.e. decreases the probability that the state evolves outside the subspace in question. This resembles the ancient Greek, Zeno paradox ($Z\acute{\eta}\nu\omega\nu$), and thus the name.

The continuum measurement limit, leads to zero probability of leaving the observed subspace. The state continues to evolve (unitarily), but restricted in the subspace of observation [4]. This implies that if we project to a one-dimensional subspace, the state stops evolving. In most literature, the question is of a particle decaying or not, so the last comment applies. In particular, the above phenomenon is still present for infinite dimensional Hilbert spaces, but provided that the restricted Hamiltonian ($H_r = PHP$) is self-adjoint, as we will see later.

3.2 Decoherent histories

Decoherent histories approach to quantum theory is an alternative formulation designed to deal with closed systems and it was developed by Griffiths [5], Omnès [6], and Gell-Mann and Hartle [7]. There is no external observer, no a-priori environment-system split. The main mathematical aim of this approach, is to see when is it meaningful to assign probabilities to a history of a closed quantum system and of course to determine this probability.

Here we will revise the standard non-relativistic quantum mechanics in decoherent histories formulation. To each history ($\underline{\alpha}$) corresponds a particular class operator $C_{\underline{\alpha}}$,

$$C_{\underline{\alpha}} = P_{\alpha_n} e^{-iH(t_n - t_{n-1})} P_{\alpha_{n-1}} \cdots e^{-iH(t_2 - t_1)} P_{\alpha_1} \tag{1}$$

Where P_{α_1} etc are projection operators corresponding to some observable, H is the Hamiltonian, and t_n is the total time interval we consider. This class operator corresponds to the history, the system is at the subspace spanned by P_{α_1} at time t_1 at P_{α_2} at time t_2 and so on. The probability for this history, provided we had some external observer making the measurement at each time t_k would be

$$p(\underline{\alpha}) = D(\underline{\alpha}, \underline{\alpha}) = \mathrm{Tr}(C_{\underline{\alpha}} \rho C^{\dagger}_{\underline{\alpha}}) \tag{2}$$

where ρ is the initial state. In the case of a closed system, Eq.(2) fails in general to be probability due to interference[2].

There are, however, certain cases where we can assign probabilities. This happens if for a complete set of histories, they pairwise obey

$$D(\underline{\alpha}, \underline{\beta}) = \mathrm{Tr}(C_{\underline{\alpha}} \rho C^{\dagger}_{\underline{\beta}}) = 0 \quad \forall \quad \underline{\alpha} \neq \underline{\beta} \tag{3}$$

In that case, the complete set of histories is called *decoherent* set of histories and we can assign to each history of this set the probability of Eq.(2). In order to achieve a set

[2]The additivity of disjoint regions of the sample space is not satisfied by Eq.(2)

of histories that satisfy Eq.(3) in general we need to consider coarse grained histories, or/and very specific initial state ρ[3].

To sum up, in decoherent histories we need to first construct a class operators that corresponds to the histories of interest[4], and then confirm that these histories satisfy Eq.(3). Only then we can give an answer.

3.3 The restricted propagator

A mathematical object that will be needed for computing the suitable class operators, is the restricted propagator. This is the propagator restricted to some particular region Δ (of the configuration space) that corresponds to a subspace of the total Hilbert space denoted by $\mathcal{H}_\Delta$. The most common (but not the most general) is the path integral definition:

$$g_r(x,t \mid x_0,t_0) = \int_\Delta \mathcal{D}x \exp(iS[x(t)]) = \langle x|g_r(t,t_0)|x_0\rangle \tag{4}$$

The integration is done over paths that remain in the region Δ during the time interval $[t,t_0]$. The $S[x(t)]$ is as usual the action. The operator form of the above is given by [8, 9]:

$$g_r(t,t_0) = \lim_{\delta t\to 0} Pe^{-iH(t_n-t_{n-1})}P\cdots Pe^{-iH(t_1-t_0)}P \tag{5}$$

With $t_n = t$, $\delta t \to 0$ and $n\to\infty$ simultaneously keeping $\delta t \times n = (t-t_0)$. H is the Hamiltonian operator. P is a projection operator on the restricted region Δ. We therefore have

$$g_r(x,t \mid x_0,t_0) = \langle x|g_r(t,t_0)|x_0\rangle \tag{6}$$

Note here that the expression Eq.(5) is the defining one for cases that the restricted region is not a region of the configuration space, but some other subspace of the total Hilbert space $\mathcal{H}$. The differential equation obeyed by the restricted propagator is:

$$(i\frac{\partial}{\partial t} - H)g_r(t,t_0) = [P,H]g_r(t,t_0) \tag{7}$$

Which is almost the Schrödinger equation, differing by the commutator of the projection to the restricted region with the Hamiltonian.

The most useful form, for our discussion was derived in Ref. [3]

$$g_r(t,t_0) = P\exp\left(-i(t-t_0)PHP\right)P \tag{8}$$

Note that PHP is the Hamiltonian projected in the subspace $\mathcal{H}_\Delta$. To prove Eq.(8) we multiply Eq. (7) with P we will then get

$$(i\frac{\partial}{\partial t} - PHP)g_r(t,t_0) = 0 \tag{9}$$

using the fact that $P[H,P]P = 0$ and that the propagator has a projection P at the final time. This is Schrödinger equation with Hamiltonian PHP. It is evident that

[3]Note that the interaction of a system with an environment that brings decoherence, in the histories vocabulary, is just a particular type of coarse graining where we ignore the environments degrees of freedom.

[4]Note that the same classical question can be turned to quantum with several, possibly inequivalent ways. Due to this property, the construction of the suitable class operator is important for questions such as for example, the arrival time or reparametrization invariant questions.

this leads to the full propagator in $\mathcal{H}_\Delta$ provided that the operator PHP is self-adjoint in this subspace [4][5].

4 Quantum Zeno histories

In this section we will examine the question *what is the probability for a system to remain in a particular subspace, during a time interval* $\Delta t = t - t_0$. We will see the probabilities and decoherence conditions for the general case, and then see what this implies for the arrival time problem, which is just a particular example.

4.1 The class operator and probabilities

There are several ways of turning the above classical proposition to a quantum mechanical one. The most straight forward is the following. We consider a system being in one subspace by projecting to that, and the history of always remaining in that subspace corresponds to the limit of projecting to the region evolving unitarily but for infinitesimal time and then projecting again, i.e. taking the δt between the propositions going to zero. The class operator for remaining always in that subspace follows from Eq.(1) by taking each $P_{\alpha k}$ being the same (P) and taking the limit of $(t_k - t_{k-1})$ going to zero for each k. We then have

$$C_\alpha(t, t_0) = g_r(t, t_0) \tag{10}$$

and the class operator for not remaining at this subspace during all the interval is naturally

$$C_\beta(t, t_0) = g(t, t_0) - g_r(t, t_0) \tag{11}$$

with $g(t, t_0) = \exp(-iH(t - t_0))$ the full propagator.

Let us, for the moment, assume that the initial state $|\psi\rangle$ is such, that we do have decoherence. We will return later to see when this is the case. The (candidate) probability is

$$p(\alpha) = \langle\psi|g_r^\dagger(t, t_0)g_r(t, t_0)|\psi\rangle \tag{12}$$

Following Eq.(8) it is clear[6] that

$$g_r^\dagger(t, t_0)g_r(t, t_0) = P \tag{13}$$

which then implies

$$p(\alpha) = \langle\psi|P|\psi\rangle \tag{14}$$

For an initial state that is in the subspace defined by P, the probability to remain in this subspace is one. This is the usual account of the quantum Zeno effect. As it is stressed in other literature, to have the quantum Zeno is crucial that the restricted Hamiltonian $H_r = PHP$ to be self-adjoint operator in the subspace. Note, that this only states that the system remain in the subspace, but it does not "freeze" completely and in particular follows unitary evolution in the subspace with Hamiltonian, the restricted

[5] A detailed proof from Eq.(5) can be found in [3].

[6] Provided PHP is self-adjoint in the subspace. This is true for finite dimensional Hilbert spaces and has been shown to be true for regions of the configuration space in a Hamiltonian with at most quadratic momenta [4].

one H_r. The form of Eq.(8) of the restricted propagator makes the latter comment more transparent.

4.2 Decoherence condition

All this is well understood for open systems with external observers. To assign the candidate probability of Eq.(12) as a proper probability of a closed system, we need the system to obey the decoherence condition, i.e.

$$D(\alpha,\beta)=\langle\psi|C_\beta^\dagger C_\alpha|\psi\rangle=0 \tag{15}$$

and this implies that

$$\langle\psi|g_r^\dagger(t,t_0)g(t,t_0)|\psi\rangle=\langle\psi|P|\psi\rangle \tag{16}$$

which is a very restrictive condition and only very few states satisfy this, as we will see in the arrival time example. The condition, essentially states that the overlap of the time evolved state $(g(t,t_0)|\psi\rangle)$ with the state evolved in the subspace $(g_r(t,t_0)|\psi\rangle)$ should be the same at the times t_0 and t. Given that the restricted Hamiltonian leads, in general, to different evolution, the condition refers only to very special initial states with symmetries, or for particular time intervals Δt.

5 Arrival time problem

The arrival time problem is the following: *What is the probability that the system crosses a particular region Δ of the configuration space, at any time during the time interval $\Delta t=(t-t_0)$.* One can attempt to answer this, by considering what is the probability that the system remains always in the complementary region $\bar{\Delta}$. So if $\mathcal{Q}$ is the total configuration space, we have $\Delta\cup\bar{\Delta}=\mathcal{Q}$ and $\Delta\cap\bar{\Delta}=\emptyset$. Taking this approach to the arrival time problem, the relation with the quantum Zeno histories is apparent, since it is just the special case, where the subspace of projection is a region of the configuration space $(\bar{\Delta})$ and the Hamiltonian is quadratic in momenta, i.e.

$$\begin{aligned}\bar{P} &= \int_{\bar{\Delta}}|x\rangle\langle x|dx\\ \hat{H} &= \hat{p}^2/2m+V(\hat{x})\end{aligned} \tag{17}$$

This particular case is infinite dimensional, but as shown in Ref. [4] the restricted Hamiltonian is indeed self-adjoint and the arguments of the previous section apply.

Before proceeding further, we should point out that one could construct different class operators that would also correspond to the (classical) arrival time question. For example, one could consider having POVM's[7] instead of projections at each moment of time, or could have a finite (but frequent) number of projections (not taking the limit where $\delta t\to 0$). These and other approaches are not discussed here.

Let us see now, what the quantum Zeno effect implies about the arrival time. It states that a system initially localized outside Δ will always remain outside Δ (if it decoheres) and therefore we can only get zero crossing probabilities. This is definitely surprising, since for a wave packet that is initially localized in $\bar{\Delta}$ and its classical trajectory crosses region Δ, we would expect to get crossing probability one. The resolution comes due to the decoherence condition as will be argued later.

[7]Positive Operator Valued Measure

Returning to the decoherence condition Eq.(16) we see that there is the overlap of the time evolved state with the restricted time evolved state. In the arrival time case, the restricted Hamiltonian corresponds to the Hamiltonian in the restricted region ($\bar{\Delta}$) but with infinite potential walls on the boundary (i.e. perfectly reflecting). We then get decoherence in the following four cases.

(a) The initial state $|\psi\rangle$ is in an energy eigenstate, and it also vanishes on the boundary of the region.

(b) The restricted propagator can be expressed by the method of images[8] and the initial state shares the same symmetry.

(c) The full unitary evolution in the time interval Δt remains in the region $\bar{\Delta}$.

(d) Recurrence: Due to the period of the Hamiltonian and the restricted Hamiltonian their overlap happens to be the same after some time t as it was in time t_0. This depends sensitively on the time interval and it is thus of less physical significance.

It is now apparent that most initial states do not satisfy any of those conditions. In particular, the wavepacket that classically would cross the region Δ, will not satisfy any of these conditions, and we would not be able to assign the candidate probability as a proper one, and thus we avoid the paradox. The introduction of an interacting environment to our system, (that usually produces decoherence by coarse-graining the environment) does not change the probabilities and contrary to the intuitive feeling, it does not provide decoherence for the particular type of question we consider. This still leave us with no answer for any of the cases that the system would classically cross the region. The latter implies, that the straight forward coarse grainings we used, were not general enough to answer fully the arrival time question [9].

As a final note, we should point out that the quantum Zeno effect in the decoherent histories, has implications for the decoherent histories approach to the problem of time (e.g. Refs. [9, 3]).

6 Conclusions

We examined the quantum Zeno type of histories of a closed system, using the decoherent histories approach. We show that the quantum Zeno effect is still present, but only for the very few cases that we have decoherence. The situation does not change with the introduction of interacting environment. We see that while in the open system quantum Zeno, the delay of the evolution arises as interaction with the observer, in the closed system we have the decoherence condition “replacing” the observer and resolving the apparent paradox.

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[8] Note that the restricted propagator can be expressed using the method of images, if and only if there exist a set of energy eigenstates, vanishing on the boundary, that when projected on the region $\bar{\Delta}$ forms a dense subset of the subspace $\mathcal{H}_{\bar{\Delta}}$, i.e. span $\mathcal{H}_{\bar{\Delta}}$. This is equivalent with requiring that the restricted energy spectrum (i.e. spectrum of the restricted Hamiltonian H_r) is a subset of the (unrestricted) energy spectrum, which is not in general the case.

[9] For more details, examples and discussion see Ref. [3].

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